$\alpha_2 \;\; = (D/(\Delta s)^2) + (1/A\Delta s)\, q$
$\alpha_3 \;\; = (D/(\Delta s)^2)$
$\alpha_4 \;\; = (-\Delta H_1/\rho C_p)$
$\alpha_5 \;\; = (-\Delta H_2/\rho C_p)$
$\alpha_6 \;\; = (-UA_w/A\rho C_p)$
$\rho \;\; =$ density
$\Psi \;\; =$ constant total concentration $= X + Y + Z$

Subscripts

o = feed condition
ss = steady state value
c = coolant condition
max = maximum permissible value
min = minimum permissible value

Superscripts

o = initial condition at $t = 0$
d = desired level of operation
x, y, z, T = associate functions or constants with X, Y, Z, and T, respectively

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Manuscript received November 8, 1965; revision received March 8, 1966; paper accepted March 9, 1966.

Generalized Couette Flow of an Ellis Fluid

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The motion of an ordinary Newtonian fluid in a channel caused by a pressure gradient alone, or caused by the motion of one of the walls, can be obtained separately; and the problem of generalized Couette flow (the motion due to both causes simultaneously) can be obtained by superposition. For an Ellis fluid, because of the nonlinearity of the viscosity-stress law, the principle of superposition is not applicable. This paper is devoted to the solution of this problem of generalized Couette flow of an Ellis fluid in parallel flat plate and circular annular channels.

Ellis fluid has been characterized by Matsuhisa and Bird (*1*) to be one for which the coefficient of viscosity η is not a constant but varies with the stress in accordance with the law

$$\frac{1}{\eta} = \frac{1}{\eta_o}\left[1 + \left|\frac{\tau}{\tau_{1/2}}\right|^{\alpha-1}\right] \tag{1}$$

where

$$\tau = \frac{1}{2}\sqrt{\sum_i \sum_j \tau_{ij}^2} \tag{2}$$

Equation (1) contains three constants η_o, α, and $\tau_{1/2}$ which must be determined experimentally for each fluid. The constant η_o can be interpreted as the viscosity at zero stress. The numerical constant α has been found for most fluids to have a value between 1 and 2. When $\alpha = 1$, Equation (1) shows that η is constant (equal to ½ η_o). We can, thus, look upon the ordinary Newtonian fluid to be a special case of the Ellis fluid for $\alpha = 1$. The case $\eta = ½\, \eta_o$ can also be looked upon as pertaining to a physical situation for which the stress τ remains constant always (equal to $\tau_{1/2}$).

Matsuhisa and Bird have obtained a few simple solutions of the fluid motion governed by the viscosity law (*1*). These solutions made use of the fact of dynamic symmetry, which simplified the analysis. When one considers a one-dimensional flow problem, the inertia terms in the momentum equation may either drop out or become linearized because of the geometry of the problem, but the final equation for the Ellis fluid still remains nonlinear. Such independent causes as the motion of one of the channel walls while the other is held stationary or the presence of a nonzero pressure gradient produce motions which for the ordinary Newtonian fluid may be superposed. For the Ellis fluid, because of the nonlinear viscosity stress law, the solution of the generalized Couette flow has to be obtained as a single composite problem. The present paper is devoted to obtaining this solution.

GOVERNING EQUATIONS

Consider the steady flow of an Ellis fluid contained between two infinite parallel plates. Let x axis be parallel to and Y axis normal to the plates. Let $Y = 0$ be the stationary plate and $Y = 2h$ the plate moving with a uniform velocity U along the x axis. The plates are infinite in the z and x directions so that the only nonvanishing component of velocity is v_x, and the only nonvanishing component of stress is τ_{xY}; and both of these are functions of x alone. With these simplifications and the dimensionless quantities

$$y = \frac{Y}{2h}, \quad u = \frac{\eta_o}{2h\,\tau_{1/2}} v_x, \quad K = \frac{\eta_o}{2h\,\tau_{1/2}} U, \quad \sigma = \frac{\tau_{xY}}{\tau_{1/2}}, \quad P = \frac{2h}{\tau_{1/2}}\frac{dp}{dx} \tag{3}$$

the equations of motion are

$$0 = -P + \frac{d\sigma}{dy} \tag{4}$$

$$\frac{du}{dy} = \sigma(1 + |\sigma|^{\alpha-1}) \tag{5}$$

The boundary conditions to be satisfied are

$$u = 0 \quad \text{at} \quad y = 0 \tag{6}$$

$$u = K \quad \text{at} \quad y = 1 \tag{7}$$

SOME OBSERVATIONS FOR THE ORDINARY VISCOUS FLUID

Before one attempts to solve the Equations (4) and (5) subject to the conditions (6) and (7), it will be of interest to make some observations for the solution of the generalized Couette flow problem for the ordinary Newtonian fluid. With $\alpha = 1$, the solution of (4) and (5) subject to conditions (6) and (7) is

$$u = Ky\,[1 + \lambda(1-y)] \tag{8}$$

$$\sigma = K\,[1 + \lambda(1-2y)]$$

where

$$\lambda = -\frac{P}{2K}$$

The following remarks can be made about Equation (8). For favorable pressure gradient (λ positive), the velocity remains positive throughout the width of the channel. The shear stress also remains positive throughout the channel width if $0 < \lambda < 1$ but changes sign somewhere in the channel if $\lambda > 1$.

For adverse pressure gradient (λ negative), if $-1 < \lambda < 0$ the velocity and the shear stress both remain positive everywhere in the channel. If $\lambda < -1$ the velocity changes sign somewhere in the channel and so does the shear stress.

SOLUTION FOR THE ELLIS FLUID

The absolute value sign in Equation (5) necessitates that the sign of σ must be ascertained before the integration of (5) can be carried out. On the other hand, until the equations are integrated and the boundary conditions used, there is no way of knowing the sign of σ. Such difficulties do not arise in the simple Couette flow or the simple Poiseuille flow owing to the following reason:

Equation (4) is quickly integrated to give

$$\sigma = Py + A \tag{9}$$

For simple Couette flow (when one wall is moving, the other stationary, and there is no pressure gradient)

$$\sigma = A$$
$$u = Ky \tag{10}$$

where

$$K = A(1 + |A|^{\alpha-1})$$

Thus, σ does not change sign anywhere in the channel. For simple Poiseuille flow (when both walls are stationary and $P \neq 0$), one can invoke the condition of symmetry about the center of the channel, namely $\sigma = 0$ at $y = ½$, and obtain $\sigma = P(y - ½)$, $½ < y < 1$. Thus, in this case also the shear stress does not change sign anywhere in the half width ($½ < y < 1$) of the channel. The solution in the other half of the channel is obtained from symmetry.

The above discussion indicates that for the general case when the pressure gradient is nonzero and one of the walls moves while the other is held stationary, we need consider separately the two cases: first when the sign of the shear stress does not change anywhere in the channel, and second when it does.

When σ Does Not Change Sign

Substituting (9) into (5), integrating it, and using condition (6), we get

$$u = \frac{Py^2}{2} + Ay + \frac{|Py + A|^{\alpha+1} - |A|^{\alpha+1}}{P(\alpha+1)} \tag{11}$$

A is the only undetermined integration constant in (11), and condition (7) gives

$$K = \frac{P}{2} + A + \frac{|P + A|^{\alpha+1} - |A|^{\alpha+1}}{P(\alpha+1)} \tag{12}$$

For fixed values of K, P, and α, the constant A can be determined from (12). However, (12) is an algebraic equation involving decimal exponents of the absolute value of $P + A$ and of A. It will be more convenient to determine the value of A as follows. We express (9), (11), and (12) in the alternate form

$$\frac{\sigma}{P} = y + \theta - ½ \tag{9a}$$

$$\frac{u}{P} = \frac{y^2}{2} + \left(\theta - \frac{1}{2}\right) y + \frac{|P|^{\alpha-1}}{\alpha+1}\left[\left|y + \theta - \frac{1}{2}\right|^{\alpha+1} - \left|\theta - \frac{1}{2}\right|^{\alpha+1}\right], \tag{11a}$$

$$\frac{K}{P} = \theta + \frac{|P|^{\alpha-1}}{\alpha+1}\left[\left|\theta + \frac{1}{2}\right|^{\alpha+1} - \left|\theta - \frac{1}{2}\right|^{\alpha+1}\right] \tag{12a}$$

where

$$\theta = \frac{1}{2} + \frac{A}{P} \tag{13}$$

Written in this form we notice that the sign of σ/P will not change anywhere in the channel if the value of θ determined from (12a) lies outside the range $-½ < \theta < ½$. Now the right side of (12a) is

$$F(\theta) = \theta + \frac{|P|^{\alpha-1}}{\alpha+1}\left(\left|\theta + \frac{1}{2}\right|^{\alpha+1} - \left|\theta - \frac{1}{2}\right|^{\alpha+1}\right) \tag{14}$$

and is seen to be an odd function of θ. For positive θ, (14) may be written as

$$F(\theta) = \theta + \frac{|P|^{\alpha-1}}{\alpha+1}\left[\left(\theta+\frac{1}{2}\right)^{\alpha+1} - \left(\frac{1}{2}-\theta\right)^{\alpha+1}\right]$$

$$0 \leq \theta \leq \frac{1}{2}$$

$$= \theta + \frac{|P|^{\alpha-1}}{\alpha+1}\left[\left(\theta+\frac{1}{2}\right)^{\alpha+1} - \left(\theta-\frac{1}{2}\right)^{\alpha+1}\right] \quad \frac{1}{2} \leq \theta \tag{15}$$

Thus $F(\theta)$ increases continuously from 0 to ∞ for positive θ, and because of the fact that $F(\theta)$ is an odd function, it increases continuously from $-\infty$ to 0 for negative θ. The graph of $F(\theta)$ vs. θ will be intersected at one point by the ordinate K/P. If this intersection occurs at a value of θ outside the range $(-\frac{1}{2} < \theta < \frac{1}{2})$, the analysis of this case is applicable, and (9*a*) and (11*a*) give the solution.

When the Shear Stress Changes Sign

If the intersection of the graph of $F(\theta)$ vs. θ by the ordinate K/P occurs at a value of θ within the range $-\frac{1}{2} < \theta < \frac{1}{2}$, the shear stress changes sign somewhere in the channel. Let σ vanish at $y = \xi$ $(0 < \xi < 1)$. Therefore from (4) and (5), we get

$$\sigma = P(y - \xi) \tag{16}$$

$$\frac{du}{dy} = P[y - \xi - |P|^{\alpha-1}(\xi - y)^{\alpha}] \quad 0 \leq y \leq \xi$$

$$= P[y - \xi + |P|^{\alpha-1}(y-\xi)^{\alpha}] \quad \xi \leq y \leq 1 \tag{17}$$

Integrating (17) we obtain

$$u = \frac{P}{2}\left[(\xi - y)^2 + \frac{2}{\alpha+1}|P|^{\alpha-1}(\xi-y)^{\alpha+1} + C\right]$$

$$0 \leq y \leq \xi$$

$$= \frac{P}{2}\left[(y-\xi)^2 + \frac{2}{\alpha+1}|P|^{\alpha-1}(y-\xi)^{\alpha+1} + C\right]$$

$$\xi \leq y \leq 1 \tag{18}$$

It may be noticed that we used the same integration constant C in the two parts of (18) in order to ensure continuity of velocity at $y = \xi$. The use of condition (6) gives

$$-C = \xi^2 + \frac{2}{\alpha+1}|P|^{\alpha-1}\xi^{\alpha+1} \tag{19}$$

Hence the velocity (18) can be written in the form

$$\frac{u}{P} = \frac{\xi^2}{2}\left(\left|1-\frac{y}{\xi}\right|^2 - 1\right) + \frac{\xi^{\alpha+1}}{\alpha+1}|P|^{\alpha-1}\left(\left|1-\frac{y}{\xi}\right|^{\alpha+1} - 1\right) \tag{20}$$

The location of the station $y = \xi$ at which the shear stress vanishes can be obtained from the application of condition (7) in (18), which gives

$$\frac{K}{P} = \frac{1-2\xi}{2} + \frac{1}{\alpha+1}|P|^{\alpha-1}[(1-\xi)^{\alpha+1} - \xi^{\alpha+1}] \tag{21}$$

Hence, the solution of the problem is obtained from the first case if the intersection of the graph of $F(\theta)$ by the ordinate K/P gives a value of θ greater than ½ or less than $-$ ½ and from the second case if instead ξ is obtainable from (21).

AXIAL FLOW IN AN ANNULUS

Let the Ellis fluid be confined between two coaxial circular cylinders. We consider the motion caused in the fluid by a nonzero axial pressure gradient and by the axial motion of the inner cylinder (radius βa) while the outer (radius a) is held stationary. The equations governing the flow in this case are

$$\frac{1}{r}\frac{d}{dr}(r\,\tau_{rz}) = \frac{\partial p}{\partial z}$$

$$\frac{1}{\eta} = \frac{1}{\eta_o}\left(1 + \left|\frac{\tau_{rz}}{\tau_{1/2}}\right|^{\alpha-1}\right) \tag{22}$$

$$\tau_{rz} = \eta\frac{dv_z}{dr}$$

Let the outer cylinder be stationary and the inner moving in the axial direction with a velocity W. We use the dimensionless quantities

$$R = \frac{r}{a}, \quad V = \frac{\eta_o}{a\,\tau_{1/2}}v_z, \quad K = \frac{\eta_o}{a\,\tau_{1/2}}W,$$

$$\sigma = \frac{\tau_{rz}}{\tau_{1/2}}, \quad P = \frac{a}{\tau_{1/2}}\frac{dp}{dz} \tag{23}$$

Then Equation (22) becomes

$$\frac{1}{R}\frac{d}{dR}(R\sigma) = P \tag{24}$$

$$\frac{dV}{dR} = \sigma\,(1 + |\sigma|^{\alpha-1}) \tag{25}$$

The above equations are to be solved subject to the conditions

$$\text{At } R = 1, \quad V = 0 \tag{26}$$

$$\text{At } R = \beta, \quad V = K \tag{27}$$

Matsuhisa and Bird have obtained solutions of (24), (25) in the special case when $P = 0$. In the other special case when $K = 0$, they have referred to a numerical solution by McEachern. However, a formal solution of the generalized Couette flow for an annulus can be obtained as follows:

Integrating (24) we get

$$\sigma = \frac{R}{2}P + \frac{A}{R} \tag{28}$$

The constant A is nonzero unless we consider the flow in a single tube (radius a) for which the solution is

$$V = -\frac{P}{4}\left[(1-R^2) + \frac{2}{\alpha+1}\left|\frac{P}{2}\right|^{\alpha-1}(1-R^{\alpha+1})\right] \tag{29}$$

For $\alpha = 1$, Equation (29) is the same as Equation (5.7) of Schlichting's book.

For flow in an annulus with zero pressure gradient we have to solve

$$\frac{dV}{dR} = \frac{A}{R}\left(1 + \frac{|A|^{\alpha-1}}{R^{\alpha-1}}\right) \tag{30}$$

Therefore, when one uses (26), the solution of (30) is

$$V = A\left[\log R - \frac{|A|^{\alpha-1}}{\alpha-1}\left(\frac{1}{R^{\alpha-1}} - 1\right)\right] \tag{31}$$

where the constant A is given by

$$K = A\left[\log\beta - \frac{|A|^{\alpha-1}}{\alpha-1}\left(\frac{1}{\beta^{\alpha-1}} - 1\right)\right] \tag{32}$$

The above two special cases of flow in an annular channel are such that the sign of the shear stress remains invariant throughout the width of the channel. For the generalized Couette flow in an annular channel we are not sure in advance whether or not σ changes sign within the channel width. Thus, we again have to obtain the solutions in the two cases separately.

When σ does not change sign anywhere in the channel, integration of (25) with (28) and (26) gives

$$V = -\frac{P}{4}(1-R^2) + A\log R \pm \int_1^R \left|\frac{RP}{2}+\frac{A}{R}\right|^{\alpha} dR \quad (33)$$

When one uses (27), the constant A can be determined from

$$K = -\frac{P}{4}(1-\beta^2) + A\log\beta \pm \int_1^{\beta} \left|\frac{R}{2}P+\frac{A}{R}\right|^{\alpha} dR \quad (34)$$

The upper or the lower signs before the integrals in (33) and (34) are to be taken when $\frac{R}{2}P + \frac{A}{R}$ is always positive or always negative throughout the channel. The shear stress which can be written in the form

$$\frac{\sigma}{P} = \frac{1}{R}\left(\frac{R^2}{2}+\frac{A}{P}\right) \quad (35)$$

will not change sign if either $\frac{A}{P} < \frac{-1}{2}$ or $\frac{A}{P} > \frac{-\beta^2}{2}$

Hence the above solution will be useful if a value of A/P conforming to these inequalities can be found to satisfy Equation (34).

When σ does change sign, let σ vanish at $R = \xi$ where $\beta < \xi < 1$. Then

$$\sigma = \frac{P}{2}\left(R-\frac{\xi^2}{R}\right) \quad (36)$$

$$\frac{dV}{dR} = \frac{P}{2}\left[\left(R-\frac{\xi^2}{R}\right) - \left|\frac{P}{2}\right|^{\alpha-1}\left(\frac{\xi^2}{R}-R\right)^{\alpha}\right], \quad \beta<R<\xi \quad (37)$$

$$\frac{dV}{dR} = \frac{P}{2}\left[R-\frac{\xi^2}{R} + \left|\frac{P}{2}\right|^{\alpha-1}\left(R-\frac{\xi^2}{R}\right)^{\alpha}\right], \quad \xi<R<1 \quad (38)$$

Integrating (37) and using condition (27) we get

$$V-K = \frac{P}{2}\left[\frac{R^2-\beta^2}{2}\right] - \xi^2\log\frac{R}{\beta} - \left|\frac{P}{2}\right|^{\alpha-1}\int_{\beta}^{R}\left(\frac{\xi^2}{R}-R\right)^{\alpha} dR, \quad \beta<R<\xi \quad (39)$$

Integrating (38) and using condition (26) we get

$$V = \frac{P}{2}\left[\frac{R^2-1}{2}\right] - \xi^2\log R + \left|\frac{P}{2}\right|^{\alpha-1}\int_1^R\left(R-\frac{\xi^2}{R}\right)^{\alpha} dR, \quad \xi<R<1 \quad (40)$$

The location ξ at which the shear stress vanishes can be found from the fact that the velocity given by (39) and (40) is continuous at $R = \xi$, which gives

$$K = \frac{P}{2}\left[\frac{\beta^2-1}{2} + \left|\frac{P}{2}\right|^{\alpha-1}\int_1^{\xi}\left(R-\frac{\xi^2}{R}\right)^{\alpha} dR + \int_{\xi}^{\beta}\left(\frac{\xi^2}{R}-R\right)^{\alpha} dR\right] \quad (41)$$

Hence the final solution in the annular channel is given by (33) if a value of A/P less than $-\frac{1}{2}$ or greater than $-\beta^2/2$ can satisfy Equation (34). If not, the solution is given by (39) and (40) where some ξ satisfying (41) can be found.

ACKNOWLEDGMENT

The facilities available at Iowa State University where the author is a visiting associate professor are gratefully acknowledged.

NOTATION

- a = radius of the outer cylinder
- A = constant of integration in Equation (9) or (28)
- h = half width of the parallel plate channel
- K = nondimensional velocity of the moving wall
- P = nondimensional pressure gradient
- r, z = radial and axial coordinates for the annular channel
- R = nondimensional radial coordinate
- u = nondimensional velocity parallel to the plates
- U = velocity of the moving plate in the parallel plate channel
- v_x = velocity component parallel to the plates
- v_z = axial velocity for the annular channel
- V = nondimensional axial velocity for the annular channel
- W = axial velocity of the inner cylinder
- x, Y = parallel and normal coordinate for the parallel plate channel
- y = nondimensional normal coordinate for the parallel plate channel

Greek Letters

- α = numerical exponent in Equation (1)
- β = radii ratio for the annular channel
- η = coefficient of viscosity
- η_o = coefficient of viscosity at zero stress
- θ = $\frac{1}{2} + A/P$
- λ = $-P/2K$
- ξ = location in the channel where the shear stress vanishes
- σ = nondimensional shear stress
- τ = stress dyadic
- τ_{ij} = stress tensor components
- $\tau_{1/2}$ = shear stress when viscosity $\eta = \frac{1}{2}\eta_o$

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Manuscript received September 29, 1965; revision received March 4, 1966; paper accepted March 30, 1966.